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Christoffel Symbols and Inertia in Flat Space-Time Theory

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Abstract

A necessary and sufficient criterion of inertia is presented, for the flat space-time theory of general frames of reference, in terms of the vanishing of some typical components of the affine connection pertaining to curvilinear coordinate systems. The physical identification of inertial forces thus arises in the context of the special theory of relativity.

1. *Introduction*

In this paper we examine the fundamental concept of inertia in the special theory of relativity, adopting for that purpose the broader scope of the principle of general covariance. It will be shown that, provided space-time is flat, the vanishing of the components $\Gamma^i_{0\mu}(x^\nu)$ of the Christoffel symbols[1] represents a necessary and sufficient condition for the system of coordinates $\{x^\nu\}$ to define an inertial frame of reference. Some consequences of this will also be briefly discussed.

It is today firmly established (even for the special theory) that we must look upon Einsteinian relativity as the geometry of space-time interpreted from the standpoint of physics [see, for instance, Synge (1969), p. 34]. Accordingly, we essentially consider special relativity as the theory of flat-space-time.[2] Indeed, following this approach, one states the geometric characterization of Minkowski space-time by requiring a vanishing Riemann-Christoffel curvature tensor everywhere on the relativistic manifold. Physically this means, of course, that one leaves out genuine gravitational phenomena.

[1] In this paper Greek indices μ, ν, λ, ... run over 0, 1, 2, 3, and Latin indices i, j, k, ... over 1, 2, 3; thus we write $(x^\mu) = (x^0, x^i)$, with their usual meaning.

[2] The name "general theory of relativity" is usually reserved for Einstein's theory of gravitation; for example, cf. Synge (1964, 1965), Tolman (1934). However, Møller (1952) considers "general relativity" as the theory of space-time (flat or curved) when handled in a generally covariant manner.

As is well known, as a consequence of flatness there exist rectangular Cartesian (namely, Galilean) coordinates, which are preferred coordinates in special relativity only because handling them is simpler; that is, the metric tensor in terms of the Galilean coordinates becomes the Minkowski metric throughout space-time. It is obvious, however, that flat space-time geometry is a generally covariant theory, for the vanishing of the Riemann-Christoffel tensor is a generally covariant feature. Thus, general transformations of coordinates are quite permissible in special relativity (as in any theory whatever) since, in effect, the principle of general covariance has a purely formal character, being devoid by itself of any physical content.[3]

The geometric, as well as the physical, simplifications afforded by the use of Galilean coordinates are naturally due to the fact that every rectilinear coordinate system[4] defines an inertial frame of reference in Minkowski space-time. We must here remark that in the present paper we do not identify the concepts of "coordinate system" and "frame of reference"; rather, Møller's subtle distinction between these two notions will be adopted throughout [cf. Møller (1952), Chap. VIII]. From this point of view, while each space-time coordinate system (satisfying some very broad causality conditions)[5] corresponds to one, and only one, physically realizable frame of reference, the converse does not hold. Indeed, we can obviously introduce an infinity of different coordinate systems attached, nevertheless, to one and the same physical frame. As we know, this is done by means of *internal transformations* of space-time coordinates, in Møller's sense [see Møller (1952), pp. 248 ff.]. As a matter of fact, it is important to remark that these internal transformations leave invariant the spatial 3-geometry determined in a particular frame by means of standard rod measurements or, equivalently, by means of standard clock and closed light ray trip experiments.[6] We finally recall the fact that the set of internal transformations of space-time coordinates, attached to a given frame of reference, form a special group of continuous transformations.

2. *A General Criterion of Inertia*

It is clear that most allowable curvilinear coordinate systems we introduce in flat space-time correspond to accelerated frames of reference, while, conversely, the study of accelerated frames requires the use of curvilinear coordinates.[7] Therefore the considerations stated in section 1 should not mask the

[3] This point was first emphasized by Kretschmann (1917); and Einstein (1918) also concurred in the remark. See also Tolman (1934), p. 168.

[4] I.e., coordinates obtained from a Galilean set by means of linear transformations.

[5] These are discussed, for instance, by Landau and Lifshitz (1962), pp. 271 ff.

[6] The formal proof is given by Møller (1952), Appendix 4, pp. 374 ff.

[7] A very exhaustive study on the accelerated frames of reference in flat space-time theory can be found in Heintzmann and Mittelstaed (1968). The literature on this issue is rather extensive; the reader not familiar with the literature may profit from papers such as Hill (1951), Romain (1963, 1964), Marsh (1965), etc. Perhaps the best single introduction is in Møller (1952), Chap. VIII.

very special role fulfilled by the Galilean coordinates in flat space-time theory. Rectilinear coordinates are clearly not the only coordinates able to define inertial frames. However, this kind of coordinates provide us with the *simplest direct criterion of inertia*, since with them (and only with them) the Christoffel symbols vanish, which means that there are no inertial forces acting on a free particle.[8]

Any kind of nonlinear coordinate system gives rise to some nonvanishing components of the affine connection, which may or may not correspond to inertial forces. This means that flat space-time curvilinear coordinates do not provide us with a direct criterion for probing mathematically the inertial, or noninertial, character of the frame they define. Furthermore, if it so happens that the frame is an accelerated frame of reference, then there is no internal transformation of coordinates able to eliminate all the components of the Christoffel symbols; i.e., inertial forces, if present, cannot be removed by means of internal transformations of coordinates. Thus, in the spirit of the principle of general covariance, the problem arises quite naturally as to how to characterize inertia in terms of curvilinear coordinates in Minkowski space-time.

Let us then search for a *general* criterion of inertia. We will first briefly recall the concept of an internal transformation. We assume S and S' to be physical reference frames, i.e., reference frames that can be realized with the aid of real bodies, defined by the systems of allowable coordinates $\{x^\mu\}$ and $\{x'^\mu\}$, respectively. We denote this fact by writing, quite generally, $S\{x^\mu\}$ and $S'\{x'^\mu\}$. In most cases, the reference frame S will be different from the frame S'. However, if the transformation of coordinates relating the system $\{x^\mu\}$ with $\{x'^\mu\}$ is of the *form of an internal transformation*, namely,

$$x'^0 = X'^0(x^0, x^j) \tag{2.1a}$$

$$x'^i = X'^i(x^j) \tag{2.1b}$$

the systems of reference S and S' are identical [cf. Møller (1952), pp. 248 ff.]. The physical meaning of such transformations is immediate: Equation (2.1a) entails an arbitrary redefinition of the synchronizations of the clocks used in the frame of reference, while equations (2.1b) correspond to the introduction of new space coordinates in that frame.

Now, we consider an inertial frame of reference I, say, with two attached systems of coordinates: $\{x^\mu\}$ and $\{x'^\mu\}$; that is, we have $I\{x^\mu\}$ and also $I\{x'^\mu\}$. Assume the set $\{x^\mu\}$ to be a Galilean set, while $\{x'^\mu\}$ is a set of curvilinear coordinates attached to I. Thus, by hypothesis, an internal transformation scheme relates both sets, as in equations (2.1), say, and its inverse:

$$\begin{aligned} x^0 &= X^0(x'^0, x'^j) \\ x^i &= X^i(x'^j) \end{aligned} \tag{2.2}$$

[8] This point has been already discussed by Gomberoff et al. (1969).

Since the set $\{x^\mu\}$ is Galilean, we have $\Gamma^\mu_{\nu\lambda}(x) = 0$, and therefore the Christoffel symbols of the second kind pertaining to the curvilinear set $\{x'^\mu\}$ are given by

$$\Gamma'^\mu_{\nu\lambda}(x') = (\partial x'^\mu/\partial x^\rho)(\partial^2 x^\rho/\partial x'^\nu \partial x'^\lambda) \tag{2.3}$$

i.e., more explicitly, the following obtains:

$$\begin{aligned} \Gamma'^0_{\nu\lambda}(x') &= (\partial x'^0/\partial x^\rho)(\partial^2 x^\rho/\partial x'^\nu \partial x'^\lambda) \\ \Gamma'^i_{0\mu}(x') &= 0 \\ \Gamma'^i_{jk}(x') &= (\partial x'^i/\partial x^m)(\partial^2 x^m/\partial x'^j \partial x'^k) \end{aligned} \tag{2.4}$$

Moreover, one readily observes that, in any system of coordinates $\{x'^\mu\}$ (not necessarily attached to an inertial frame) the $\Gamma'^i_{0\mu}(X')$ components of the gamma symbols transform, under an internal transformation of coordinates

$$\begin{aligned} x''^0 &= X''^0(x'^0, x'^j) \\ x''^i &= X''^i(x'^j) \end{aligned} \tag{2.5}$$

say, according to the following rule:

$$\Gamma''^i_{0\mu}(x'') = (\partial x''^i/\partial x'^j)(\partial x'^0/\partial x''^0)(\partial x'^\nu/\partial x''^\mu)\Gamma'^j_{0\nu}(x') \tag{2.6}$$

Thus we have shown that, for an inertial frame of reference, we necessarily have

$$\Gamma'^i_{0\mu}(x') = 0 \tag{2.7}$$

quite generally, i.e., whatever coordinate system $\{x'^\mu\}$ we may attach to the inertial frame. This is clear, because according to equation (2.6) conditions (2.7) correspond to an invariant property under internal transformations.

We next tackle the converse problem. Let us assume that in Minkowski space-time we are given a set $\{x'^\mu\}$ of allowable curvilinear coordinates. Let then $I\{x^\mu\}$ be an inertial frame with an attached Galilean system of coordinates, and consider the transformations

$$\begin{aligned} x^\mu &= X^\mu(x'^\nu) \\ x'^\mu &= X'^\mu(x^\nu) \end{aligned} \tag{2.8}$$

We will prove that, provided conditions (2.7) hold, the set $\{x'^\mu\}$ defines indeed an inertial frame of reference I' (not necessarily identical to the frame I).

Since the set $\{x^\mu\}$ is Galilean, we have [as in equation (2.3)]

$$\Gamma'^i_{0\mu}(x') = (\partial x'^i/\partial x^\nu)(\partial^2 x^\nu/\partial x'^0 \partial x'^\mu) = 0 \tag{2.9}$$

by hypothesis. We observe that every transformation of coordinates, like (2.8) for instance, may be factorized into a two-step process; namely, we first perform a "time-preserving" transformation, of the form

$$\begin{aligned} \hat{x}^0 &= x^0 \\ \hat{x}^i &= X'^i(x^0, x^j) \end{aligned} \tag{2.10}$$

for which the inverse scheme is, say,

$$\begin{aligned} x^0 &= \hat{x}^0 \\ x^i &= \hat{X}^i(\hat{x}^0, \hat{x}^j) \end{aligned} \tag{2.11}$$

and then we proceed with a "space-preserving" transformation, i.e.,

$$\begin{aligned} x'^0 &= \hat{X}^0(\hat{x}^0, \hat{x}^j) \\ x'^i &= \hat{x}^i \end{aligned} \tag{2.12}$$

where we properly define

$$\hat{X}^0(\hat{x}^0, \hat{x}^j) = X'^0(\hat{x}^0, \hat{X}^j(\hat{x}^0, \hat{x}^j)) \tag{2.13}$$

Therefore, since (2.12) is an internal transformation, we use the general rule stated in equation (2.6), and we get

$$\hat{\Gamma}^i_{0\mu}(\hat{x}) = (\partial\hat{x}^i/\partial x'^j)(\partial x'^0/\partial\hat{x}^0)(\partial x'^\nu/\partial\hat{x}^\mu)\Gamma'^j_{0\nu}(x') = 0 \tag{2.14}$$

by hypothesis. On the other hand, using the transformation scheme stated in equations (2.10) and (2.11), the following obtains:

$$\hat{\Gamma}^i_{0\mu}(\hat{x}) = (\partial\hat{x}^i/\partial x^\nu)(\partial^2 x^\nu/\partial\hat{x}^0\partial\hat{x}^\mu) = (\partial\hat{x}^i/\partial x^j)(\partial^2 x^j/\partial\hat{x}^0\partial\hat{x}^\mu) \tag{2.15}$$

since the x^μ's are Galilean. Hence, from (2.14) and (2.15), taking into account the Jacobian of (2.10), we end up with

$$\partial^2 x^j/\partial\hat{x}^0\partial\hat{x}^\mu = 0 \tag{2.16}$$

i.e., transformation (2.11) is necessarily of the form

$$\begin{aligned} x^0 &= \hat{x}^0 \\ x^i &= C^i\hat{x}^0 + f^i(\hat{x}^j) \end{aligned} \tag{2.17}$$

where the C's are three arbitrary constants and the f's are three arbitrary time-independent functions. Clearly, the velocities of the reference points pertaining to the set $\{\hat{x}^\mu\}$, relative to the inertial frame I, are given by

$$\left.\frac{dx^i}{dx^0}\right|_{d\hat{x}^j=0} = \frac{v^i}{c} = C^i \tag{2.18}$$

which shows that the coordinate systems $\{\hat{x}^\mu\}$ and, thus, $\{x'^\mu\}$ are necessarily attached to an inertial frame of reference.

3. *Conclusions*

In the previous section we have shown, in the context of the theory of special relativity, the following physical identification to hold:

$$S\{x^\mu\} \in \text{inertial frame} \Leftrightarrow \Gamma^i_{0\mu}(x) = 0 \tag{3.1}$$

whatever space-time coordinates we may attach to the inertial reference frame.

Since (3.1) is an equivalence, it is clear that the inertial forces presented in an accelerated frame arise entirely from the nonvanishing $\Gamma^{i}_{0\mu}$ components of the Christoffel symbols. It is also clear that, when working with curvilinear coordinates in an inertial frame, the components $\Gamma^{0}_{\mu\nu}$ are nothing but "clock artifacts," while the components Γ^{i}_{jk} correspond, in this case, precisely to the affine connection of the space metric. One immediately visualizes the following property of the inertial frames: In this kind of frame the *space track* of the space-time geodesic lines corresponds to geodesics in the space 3-geometry. Moreover, using result (3.1), one readily concludes that this property is an exclusive characteristic of the inertial frames of reference.

We also wish here to observe that the priviliged role played by the Poincaré group in special relativity becomes clarified by the kind of considerations discussed in this paper. It is clear that Lorentz transformations are not the only allowable transformations of coordinates connecting two inertial frames.[9] It is only when both frames are specified in the simplest manner, i.e., by means of Galilean coordinates, that the Lorentz transformations appear on the scene. Thus, from the standpoint of inertia, we conclude that the Lorentz transformations have a twofold meaning: [10] On the one hand, Lorentz transformations describe a Galilean system of coordinates moving with uniform velocity relative to a given Galilean system, while, on the other hand, they represent the transformation law of length and time interval measurements performed by means of standard tools.

We conclude with the remark that one should not undervalue the significance of the result (3.1) if one is interested at all in drawing meaningful physical conclusions, while studying the *accelerated* frames of reference by means of the generally covariant tensor approach to flat space-time geometry.[11]

References

Einstein, A. (1918). *Annalen der Physik*, **55**, 241.

Gomberoff, L., Krause, J., and López, C. A. (1969). *American Journal of Physics.*, **37**, 1040.

Heintzmann, H., and Mittelstaedt, P. (1968). *Springer Tracts in Modern Physics*, **47**, 185.

Hill, E. L. (1951). *Physical Review*, **84**, 1165.

Krause, J. (1975). *International Journal of Theoretical Physics*, **12**, No. 1, 35.

Kretschmann, E. (1917). *Annalen der Physik* **53**, 575.

Landau, L. D., and Lifshitz, E. M. (1962). *The Classical Theory of Fields*, Pergamon Press, Oxford, 2nd Edn.

Marsh, L. McL. (1965). *American Journal of Physics.*, **33**, 934.

[9] The most general transformation connecting two inertial frames is of the form $x'^0 = X'^0(M_\nu{}^\mu x^\nu)$, $x'^i = X'^i(M_\nu{}^j x^\nu)$, where $M_\nu{}^\mu$ is a constant matrix and $\{x^\mu\}$ is a Galilean set.

[10] An enlarged role of Lorentz transformations has been stressed recently by the author, from the point of view of kinematics; cf. Krause (1975).

[11] Cf. Møller (1952), Chap. VIII, and Heintzmann et al. (1968).

Møller, C. (1952). *The Theory of Relativity*, Clarendon Press, Oxford.
Romain, J. E. (1963). *Reviews of Modern Physics.*, **35**, 376.
Romain, J. E. (1964). *Nuovo Cimento*, **33**, 1576.
Synge, J. L. (1964). *Relativity: The General Theory*, North-Holland Publishing Co., Amsterdam.
Synge, J. L. (1965). *Relativity: The Special Theory*, North-Holland Publishing Co., Amsterdam, 2nd Edn.
Tolman, R. C. (1934). *Relativity, Thermodynamics, and Cosmology*, Clarendon Press, Oxford.